COHOMOLOGY OF DRINFEL'D ALGEBRAS: A GENERAL NONSENSE APPROACH

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1. Preliminaries

Recall that a *Drinfel'd algebra* (or a *quasi-bialgebra* in the original terminology of [1]) is an object $A = (V, \cdot\,, \Delta, \Phi)$, where $(V, \cdot\,, \Delta)$ is an associative, not necessarily coassociative, unital and counital $\mathbf{k}$-bialgebra, Φ is an invertible element of $V^{\otimes 3}$, and the usual coassociativity property is replaced by the condition which we shall refer to as quasi-coassociativity:

$$(\mathbb{1} \otimes \Delta)\Delta \cdot \Phi = \Phi \cdot (\Delta \otimes \mathbb{1})\Delta, \tag{1}$$

where we use the dot $\cdot$ to indicate both the (associative) multiplication on V and the induced multiplication on $V^{\otimes 3}$. Moreover, the validity of the following "pentagon identity" is required:

$$(\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) = (1 \otimes \Phi) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \cdot (\Phi \otimes 1),$$

$1 \in V$ being the unit element and $\mathbb{1}$, the identity map on V. If $\epsilon : V \to \mathbf{k}$ ($\mathbf{k}$ being the ground field) is the counit of the coalgebra (V, Δ) then, by definition, $(\epsilon \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \epsilon)\Delta = \mathbb{1}$. We have a natural splitting $V = \overline{V} \oplus \mathbf{k}$, $\overline{V} := \mathrm{Ker}(\epsilon)$, given by the embedding $\mathbf{k} \to V$, $\mathbf{k} \ni c \mapsto c \cdot 1 \in V$.

For a $(V, \cdot\,)$-bimodule N, recall the following generalization of the *M-construction* of [5, par. 3] introduced in [4]. Let $F^* = \bigoplus_{n \geq 0} F^n$ be the free unitary nonassociative $\mathbf{k}$-algebra generated by N, graded by the length of words. The space F^n is the direct sum of copies of $N^{\otimes n}$ over the set Br_n of full bracketings of n symbols, $F^n = \bigoplus_{u \in \mathrm{Br}_n} N_u^{\otimes n}$. For example, $F^0 = \mathbf{k}$, $F^1 = N$, $F^2 = N^{\otimes 2}$, $F^3 = N^{\otimes 3}_{(\bullet\bullet)\bullet} \oplus N^{\otimes 3}_{\bullet(\bullet\bullet)}$, etc. The algebra F^* admits a natural left action, $(a, f) \mapsto a \bullet f$, of the algebra $(V, \cdot\,)$ given by the rules:

(i) on $F^0 = \mathbf{k}$, the action is given by the augmentation ϵ,

(ii) on $F^1 = N$, the action is given by the left action of V on N and

(iii) $a \bullet (f \star g) = \sum (\Delta'(a) \bullet f) \star (\Delta''(a) \bullet g)$,

*Partially supported by the grant AV ČR #119 105 and by the Gelbart Research Institute, Bar Ilan University, Israel

where $\star$ stands for the multiplication in F^* and we use the Sweedler notation $\Delta(a) = \sum \Delta'(a) \otimes \Delta''(a)$. The right action $(f, b) \mapsto f \bullet b$ is defined by similar rules. These actions define on F^* the structure of a $(V, \cdot\,)$-bimodule.

Let $\sim$ be the relation on F^* $\star$-multiplicatively generated by the expressions of the form

$$\sum \Big((\Phi_1 \bullet x) \star \big((\Phi_2 \bullet y) \star (\Phi_3 \bullet z) \big) \Big) \sim \sum \Big(\big((x \bullet \Phi_1) \star (y \bullet \Phi_2) \big) \star (z \bullet \Phi_3) \Big),$$

where $\Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ and $x, y, z \in F^*$. Put $\bigodot(N) := F/\sim$. Just as in [5, Proposition 3.2] one proves that the $\bullet$-action induces on $\bigodot(N)$ the structure of a $(V, \cdot\,)$-bimodule (denoted again by $\bullet$) and that $\star$ induces on $\bigodot(N)$ a nonassociative multiplication denoted by $\odot$. The operations are related by

$$a \bullet (f \odot g) = \sum \left(\Delta'(a) \bullet f\right) \odot \left(\Delta''(a) \bullet g\right) \text{ and } (f \odot g) \bullet b = \sum \left(f \bullet \Delta'(b)\right) \odot \left(g \bullet \Delta''(b)\right),$$

for $a, b \in V$ and $f, g \in \bigodot(N)$. The multiplication $\odot$ is quasi-associative in the sense

$$\sum (\Phi_1 \bullet x) \odot \big((\Phi_2 \bullet y) \odot (\Phi_3 \bullet z) \big) = \sum \big((x \bullet \Phi_1) \odot (y \bullet \Phi_2) \big) \odot (z \bullet \Phi_3) \tag{2}$$

The construction described above is functorial in the sense that any $(V \cdot\,)$-bimodule map $f : N' \to N''$ induces a natural $(V \cdot\,)$-linear algebra homomorphism $\bigodot(f) : \bigodot(N') \to \bigodot(N'')$. As it was shown in [5], for any $(V, \cdot\,)$-bimodule N there exists a natural homomorphism of $\mathbf{k}$-modules $J = J(N) : \bigodot(N) \to \bigotimes(N)$. If $f : N' \to N''$ is as above then $J(N'') \circ \bigodot(f) = \bigotimes(f) \circ J(N')$.

Since the defining relations (2) are homogeneous with respect to length, the grading of F^* induces on $\bigodot(N)$ the grading $\bigodot^*(N) = \bigoplus_{i \geq 0} \bigodot^i(N)$. If N itself is a graded vector space, we have also the obvious second grading, $\bigodot(N) = \bigoplus_j \bigodot(N)^j$, which coincides with the first grading if N is concentrated in degree 1.

Let $\mathrm{Der}^n_V(\bigodot(N))$ denote the set of $(V, \cdot\,)$-linear derivations of degree n (relative to the second grading) of the (nonassociative) graded algebra $\bigodot(N)^*$. One sees immediately that there is an one-to-one correspondence between the elements $\theta \in \mathrm{Der}^n_V(\bigodot N)$ and $(V, \cdot\,)$-linear homogeneous degree n maps $f : N^* \to \bigodot(N)^*$.

If $N = X \oplus Y$, then $\bigodot(X \oplus Y)$ is naturally bigraded, $\bigodot^{*,*}(X \oplus Y) = \bigoplus_{i,j \geq 0} \bigodot^{i,j}(X \oplus Y)$, the bigrading being defined by saying that a monomial w belongs to $\bigodot^{i,j}(X \oplus Y)$ if there are exactly i (resp. j) occurrences of the elements of X (resp. Y) in w. If X, Y are graded vector spaces then there is a second bigrading $\bigodot(X \oplus Y)^{*,*} = \bigoplus_{i,j} \bigodot(X \oplus Y)^{i,j}$ just as above.

Let $(\mathcal{B}_*(V), d_{\mathcal{B}})$ be the (two-sided) normalized bar resolution of the algebra $(V, \cdot\,)$ (see [2, Chapter X]), but considered with the opposite grading. This means that $\mathcal{B}_*(V)$ is the graded $(V \cdot\,)$-bimodule, $\mathcal{B}_*(V) = \bigoplus_{n \leq 1} \mathcal{B}_n(V)$, where $\mathcal{B}_1(V) := V$ with the $(V, \cdot\,)$-bimodule structure induced by the multiplication $\cdot$, $\mathcal{B}_0(V) := V \otimes V$ (the free $(V, \cdot\,)$-bimodule on $\mathbf{k}$), and for $n \leq -1$, $\mathcal{B}_n(V)$ is the free $(V, \cdot\,)$-bimodule on $\overline{V}^{\otimes(-n)}$, i.e. the vector space $V \otimes \overline{V}^{\otimes(-n)} \otimes V$ with

the action of $(V,\cdot\,)$ given by

$u\cdot(a_0\otimes\cdots\otimes a_{-n+1}) := (u\cdot a_0\otimes\cdots\otimes a_{-n+1})$ and $(a_0\otimes\cdots\otimes a_{-n+1})\cdot w := (a_0\otimes\cdots\otimes a_{-n+1}\cdot w)$

for $u,v,a_0,a_{-n+1}\in V$ and $a_1,\ldots,a_{-n}\in\overline{V}$. If we use the more compact notation (though a nonstandard one), writing $(a_0|\cdots|a_{-n+1})$ instead of $a_0\otimes\cdots\otimes a_{-n+1}$, the differential $d_{\mathcal{B}}:\mathcal{B}_n(V)\to\mathcal{B}_{n+1}(V)$ is, for $n\leq 0$, defined as

$$d_{\mathcal{B}}(a_0|\cdots|a_{-n+1}) := \sum_{0\leq i\leq -n}(-1)^i(a_0|\cdots|a_i\cdot a_{i+1}|\cdots|a_{-n+1}).$$

Here, as is usual in this context, we make no distinction between the elements of $V/\mathbf{k}\cdot 1$ and their representatives in $\overline{V}$. We use the same convention throughout all the paper. Notice that the differential $d_{\mathcal{B}}$ is a $(V,\cdot\,)$-bimodule map.

Put $\bigodot(V,\mathcal{B}_*(V)) := \bigodot(\uparrow V\oplus\uparrow\mathcal{B}_*(V))$, where $\uparrow$ denotes, as usual, the suspension of a graded vector space and V is interpreted as a graded vector space concentrated in degree zero. Let $\mathrm{Der}^i_V(\bigodot(V,\mathcal{B}_*(V)))$ denote, for each i, the space of degree i derivations of the algebra $\bigodot(V,\mathcal{B}_*(V))$ which are also $(V,\cdot\,)$-linear maps. Let us define the derivation $D_{-1}\in\mathrm{Der}^1_V(\bigodot(V,\mathcal{B}_*(V)))$ by $D_{-1}|_{\uparrow\mathcal{B}_*(V)} :=\uparrow d_{\mathcal{B}}\downarrow$ and $D_{-1}|_{\uparrow V} := 0$. Clearly $D_{-1}(\bigodot(V,\mathcal{B}_*(V))^{i,j})\subset\bigodot(V,\mathcal{B}_*(V))^{i,j+1}$ and $D_{-1}(\bigodot^{i,j}(V,\mathcal{B}_*(V)))\subset\bigodot^{i,j}(V,\mathcal{B}_*(V))$ for any $i,j\geq 0$. We also see immediately that $D_{-1}^2=0$.

Let us consider, for any $n\geq 1$, the complex $(\bigodot^{n-1,1}(V,\mathcal{B}_*(V)),D_{-1})$, i.e. the complex

$$0\longleftarrow\bigodot{}^{n-1,1}(V,V)\xleftarrow{D_{-1}}\bigodot{}^{n-1,1}(V,\mathcal{B}_0(V))\xleftarrow{D_{-1}}\bigodot{}^{n-1,1}(V,\mathcal{B}_{-1}(V))\longleftarrow\cdots$$

Lemma 1.1. *The complex $(\bigodot^{n-1,1}(V,\mathcal{B}_*(V)),D_{-1})$ is acyclic, for any $n\geq 1$.*

Proof. We have the decomposition

$$\bigodot{}^{n-1,1}(V,\mathcal{B}_*(V)) = \bigoplus_{1\leq i\leq n}\bigodot{}^{n-1,1}_i(V,\mathcal{B}_*(V)),$$

where $\bigodot^{n-1,1}_i(V,\mathcal{B}_*(V))$ denotes the subspace of $\bigodot^{n-1,1}(V,\mathcal{B}_*(V))$ spanned by monomials having an element of $\mathcal{B}_*(V)$ at the i-th place. The differential D_{-1} obviously respects this decomposition and the canonical isomorphism J of [5] mentioned above identifies $\bigodot^{n-1,1}_i(V,\mathcal{B}_*(V))$ to $V^{\otimes(i-1)}\otimes\mathcal{B}_*(V)\otimes V^{\otimes(n-1)}$. Under this identification the differential D_{-1} coincides with $\mathbb{1}^{\otimes(i-1)}\otimes d_{\mathcal{B}}\otimes\mathbb{1}^{\otimes(n-i)}$ and the rest follows from the Künneth formula and the acyclicity of $(\mathcal{B}_*(V),d_{\mathcal{B}})$. Q.E.D.

2. Properties of $\mathrm{Der}^*_V(\bigodot(V,\mathcal{B}_*(V)))$

Let $C=(C,\cdot,1_C)$ be a unital associative algebra and let $C\xleftarrow{\epsilon}(\mathcal{R},d_{\mathcal{R}})$, $(\mathcal{R},d_{\mathcal{R}})=R_0\xleftarrow{d_{\mathcal{R}}}R_1\xleftarrow{d_{\mathcal{R}}}\cdots$, be a complex of free C-bimodules (we consider C as a C-bimodule with the bimodule

structure induced by the multiplication). Similarly, let $D \xleftarrow{\eta} (\mathcal{S}, d_\mathcal{S})$ with $(\mathcal{S}, d_\mathcal{S}) = S_0 \xleftarrow{d_\mathcal{S}} S_1 \xleftarrow{d_\mathcal{S}} \cdots$, be an acyclic complex of C-bimodules. To simplify the notation, we write sometimes R_{-1} (resp. S_{-1}, resp. $d_\mathcal{R}$, resp. $d_\mathcal{S}$) instead of C (resp. D, resp. ϵ, resp. η). Let

$$Z := \{f = (f_i)_{i \geq -1};\ f_i : R_i \to S_i \text{ a } C\text{-bimodule map and } f_i \circ d_\mathcal{R} = d_\mathcal{S} \circ f_{i+1} \text{ for any } i \geq -1\}.$$

Let us define, for a sequence $\chi = (\chi_i)_{i \geq -1}$ of C-bimodule maps $\chi_i : R_i \to S_{i+1}$, $\nabla(\chi) = (\nabla(\chi)_i)_{i \geq -1} \in Z$ by $\nabla(\chi)_i := d_\mathcal{S} \circ \chi_i + \chi_{i-1} \circ d_\mathcal{R}$. Let $B := \mathrm{Im}(\nabla) \subset Z$. For a C-bimodule M let M_I denote the set of invariant elements of M, $M_I := \{x \in M;\ cx = xc \text{ for any } c \in C\}$.

Lemma 2.1. *Under the notation above, the correspondence $Z \ni f = (f_i)_{i \geq -1} \mapsto f_{-1}(1_C) \in D_I$ induces an isomorphism $\Omega : Z/B \cong D_I/\eta(S_{0I})$. Moreover, if $f_{-1}(1_C) = \eta(h)$ for some $h \in S_{0I}$ then $f = \nabla(\chi)$ for some $\chi = (\chi_i)_{i \geq -1}$ with $\chi_{-1}(1) = h$.*

Proof. We show first that Ω is well-defined. If $f = \nabla(\chi)$ then $f_{-1} = \eta \circ \chi_{-1}$, therefore $f_{-1}(1_C) = \eta(h)$ with $h := \chi_{-1}(1_C) \in S_{0I}$ and $\Omega(f) = 0$.

Let us prove that Ω is an epimorphism. For $z \in D_I$ define a C-bimodule map $f_{-1} : C \to D$ by $f_{-1}(c) := cz$ $(= zc)$ for $c \in C$. Because $(\mathcal{R}, d_\mathcal{R})$ is free and $(\mathcal{S}, d_\mathcal{S})$ is acyclic, f_{-1} lifts to some $f = (f_i)_{i \geq -1} \in Z$ by standard homological arguments [2, Theorem III.6.1].

It remains to prove that Ω is a monomorphism. For $f = (f_i)_{i \geq -1} \in Z$, $\Omega(f) = 0$ means that $f_{-1}(1_C) = \eta(h)$ for some $h \in S_{0I}$. The C-bimodule map $\chi_{-1} : C \to S_0$ defined by $\chi_{-1}(c) := ch$ $(= hc)$ for $c \in C$ clearly satisfies $f_{-1} = \eta \circ \chi_{-1}$. A standard homological argument (see again [2, Theorem III.6.1] then enables one to extend χ_{-1} to a 'contracting homotopy' $\chi = (\chi_i)_{i \geq -1}$ with $f = \nabla(\chi)$. Q.E.D.

Definition 2.2. *For $n \geq 2$ and $k \geq 0$ let $J_k(n)$ be the subspace of* $\mathrm{Der}_V^{n-1-k}(\bigodot(V, \mathcal{B}_*(V)))$ *consisting of derivations θ satisfying*

(i) $\theta(\bigodot(V, \mathcal{B}_*(V))^{i,j}) \subset \bigodot(V, \mathcal{B}_*(V))^{i+n-1, j-k}$,

(ii) $\theta(\bigodot^{i,j}(V, \mathcal{B}_*(V))) \subset \bigodot^{i+n-1,j}(V, \mathcal{B}_*(V))$,

(iii) $[D_{-1}, \theta] = 0$ *if* $k = 0$ *and* $\theta|_{\uparrow \mathcal{B}_1(V)} = 0$ *if* $k \geq 1$.

Let us observe that, for $\theta \in J_{\geq 1}(n)$, $\theta|_{\uparrow V} = 0$. This follows from item (i) of the definition above. Observe also that $J_*(n)$ is invariant under the differential ∇ defined by $\nabla(\theta) := [D_{-1}, \theta]$, $\nabla(J_k(n)) \subset J_{k-1}(n)$ for $k \geq 1$ and $\nabla(J_0(n)) = 0$.

Proposition 2.3. *$H_{\geq 1}(J_*(n), \nabla) = 0$ while*

$$H_0(J_*(n), \nabla) = \bigodot^{n-1,1}(V, \mathcal{B}_1(V))_I \oplus \bigodot^n(V)_I. \tag{3}$$

Proof. Let $k>0$ and let $\theta \in J_k(n)$. As $\theta|_{\uparrow V}=0$, θ is given by its restriction to $\mathcal{B}_*(V)$, namely by a sequence of $(V,\cdot\,)$-bimodule maps $\theta_i : \mathcal{B}_i(V) \to \bigodot^{n-1,1}(V,\mathcal{B}_{i-k}(V))$, $i\leq 1$. Suppose that θ is a ∇-cocycle, i.e. that $\nabla(\theta)=0$. This means that the diagram

$$\begin{tikzcd}
0 & \mathcal{B}_1(V)=V \arrow[l] \arrow[d, "\theta_1"] & \mathcal{B}_0(V) \arrow[l, "d_{\mathcal{B}}"'] \arrow[d, "\theta_0"] & \mathcal{B}_{-1}(V) \arrow[l, "d_{\mathcal{B}}"'] \arrow[d, "\theta_{-1}"] & \cdots \arrow[l, "d_{\mathcal{B}}"'] \\
0 & \bigodot^{n-1,1}(V,\mathcal{B}_{1-k}(V)) \arrow[l] & \bigodot^{n-1,1}(V,\mathcal{B}_{-k}(V)) \arrow[l, "D_{-1}"'] & \bigodot^{n-1,1}(V,\mathcal{B}_{-1-k}(V)) \arrow[l, "D_{-1}"'] & \cdots \arrow[l, "D_{-1}"']
\end{tikzcd}$$

is commutative. Since $\theta_1=0$ by item (iii) of Definition 2.2, Lemma 2.1 (with $C=(V,\cdot\,,1_V)$ and $D=\bigodot^{n-1,1}(V,\mathcal{B}_{1-k}(V)))$ gives a sequence $\chi_i : \mathcal{B}_i(V) \to \bigodot^{n-1,1}(V,\mathcal{B}_{i-k-1}(V))$ of $(V,\cdot\,)$-bimodule maps, $i\leq 1$, such that $\theta_i = D_{-1}\circ\chi_i + \chi_{i+1}\circ d_{\mathcal{B}}$. We can, moreover, suppose that $\chi_1=0$, thus the sequence $(\chi_i)_{i\leq 1}$ determines a derivation $\chi\in J_{k+1}(n)$ with $\nabla(\chi)=\theta$. This proves $H_k(J_*(n),\nabla)=0$ for $k\geq 1$.

A derivation $\theta\in J_0(n)$ is given by two independent data: by the restriction $\theta_V := \theta|_{\uparrow V} : V \to \bigodot^n(V)$ and by the restriction $\theta_{\mathcal{B}_*(V)} := \theta|_{\uparrow\mathcal{B}_*(V)} : \mathcal{B}_*(V) \to \bigodot^{n-1,1}(V,\mathcal{B}_*(V))$. As $D_{-1}|_{\uparrow V}=0$, the condition $\nabla(\theta)=[D_{-1},\theta]=0$ imposes no restrictions on θ_V and, because $\nabla(\chi)|_{\uparrow V}=0$ for any $\chi\in J_1(n)$, the contribution of θ_V to $H_0(J_*(n),\nabla)$ is parametrized by $\theta_V(1_V)$, i.e. by an element of $\bigodot^n(V)_I$. This explains the second summand in (3).

The restriction $\theta_{\mathcal{B}_*(V)}$ is in fact a sequence $\theta_i : \mathcal{B}_i(V) \to \bigodot^{n-1,1}(V,\mathcal{B}_i(V))$, $i\leq 1$, of $(V,\cdot\,)$-bimodule maps and the condition $\nabla(\theta)=0$ means that the diagram

$$\begin{tikzcd}
0 & \mathcal{B}_1(V)=V \arrow[l] \arrow[d, "\theta_1"] & \mathcal{B}_0(V) \arrow[l, "d_{\mathcal{B}}"'] \arrow[d, "\theta_0"] & \mathcal{B}_{-1}(V) \arrow[l, "d_{\mathcal{B}}"'] \arrow[d, "\theta_{-1}"] & \cdots \arrow[l, "d_{\mathcal{B}}"'] \\
0 & \bigodot^{n-1,1}(V,\mathcal{B}_{1}(V)) \arrow[l] & \bigodot^{n-1,1}(V,\mathcal{B}_{0}(V)) \arrow[l, "D_{-1}"'] & \bigodot^{n-1,1}(V,\mathcal{B}_{-1}(V)) \arrow[l, "D_{-1}"'] & \cdots \arrow[l, "D_{-1}"']
\end{tikzcd}$$

is commutative. Similarly as above, a derivation $\chi\in J_1(n)$ is given by a sequence $\chi_i : \mathcal{B}_i(V) \to \bigodot^{n-1,1}(V,\mathcal{B}_{i-1}(V))$, $i\leq 1$, of $(V,\cdot\,)$-linear maps. The condition $\nabla(\chi)=\theta$ then means that $\theta_i = D_{-1}\circ\chi_i + \chi_{i+1}\circ d_{\mathcal{B}}$, especially, $\theta_1 = D_{-1}\circ\chi_1$. This last equation implies, since $\chi_1=0$ by (iii) of Definition 2.2, that $\nabla(\chi)=\theta$ forces $\theta_1(1_V)=0$. On the other hand, if $\theta_1(1_V)=0$ then Lemma 2.1 gives a $\chi\in J_1(n)$ with $\theta=\nabla(\chi)$ and we conclude that the contribution of $\theta_{\mathcal{B}_*(V)}$ to $H_0(J_*(n),\nabla)$ is parametrized by $\theta_{\mathcal{B}_*(V)}(1_V)\in\bigodot^{n-1,1}(V,\mathcal{B}_1(V))_I$ which is the first summand of (3). Q.E.D.

Let us recall that a (right) *differential graded (dg) comp algebra* (or a *nonunital operad* in the terminology of [3]) is a bigraded differential space $L = (L_*(*), \nabla)$, $L_*(*) = \bigoplus_{k\geq 0, n\geq 2} L_k(n)$, $\nabla(L_k(n)) \subset L_{k-1}(n)$, together with a system of bilinear operations

$$\circ_i : L_p(a) \otimes L_q(b) \to L_{p+q}(a+b-1)$$

given for any $1 \leq i \leq b$ such that, for $\phi \in L_p(a)$, $\psi \in L_q(b)$ and $\nu \in L_r(c)$,

$$\phi \circ_i (\psi \circ_j \nu) = \begin{cases} (-1)^{p\cdot q} \cdot \psi \circ_{j+a-1} (\phi \circ_i \nu), & \text{for } 1 \leq i \leq j-1, \\ (\phi \circ_{i-j+1} \psi) \circ_j \nu, & \text{for } j \leq i \leq b+j-1, \text{ and} \\ (-1)^{p\cdot q} \cdot \psi \circ_j (\phi \circ_{i-b+1} \nu), & \text{for } i \geq j+b. \end{cases} \tag{4}$$

We suppose, moreover, that for any $\phi \in L_p(a)$, and $\psi \in L_q(b)$, $1 \leq i \leq b$,

$$\nabla(\phi \circ_t \psi) = \nabla(\phi) \circ_t \psi + (-1)^p \cdot \phi \circ_t \nabla(\psi).$$

Any dg comp algebra determines a nonsymmetric (unital) operad in the monoidal category of differential graded spaces (see [6] for the terminology). To be more precise, let $L = (L_*(*), \circ_i, \nabla)$ be a dg comp algebra as above and let us define the bigraded vector space $\mathcal{L}_*(*) = \bigoplus_{k\geq 0, n\geq 1}$ by $\mathcal{L}_*(n) := L_*(n)$ for $n \geq 2$ and $\mathcal{L}_*(1) = \mathcal{L}_0(1) := \mathrm{Span}(1_{\mathcal{L}})$, where $1_{\mathcal{L}}$ is a degree zero generator. Let us extend the definition of structure maps $\circ_i$ to $\mathcal{L}$ by putting $f \circ_1 1_{\mathcal{L}} := f$ and $1_{\mathcal{L}} \circ_i g := g$, for $f \in \mathcal{L}_*(m)$, $g \in \mathcal{L}_*(n)$ and $1 \leq i \leq n$. Let us extend the differential ∇ by $\nabla(1_{\mathcal{L}}) := 0$. In [3] we proved the following proposition.

Proposition 2.4. *The composition maps* $\gamma : \mathcal{L}_*(a) \otimes \mathcal{L}_*(n_1) \otimes \cdots \otimes \mathcal{L}_*(n_a) \to \mathcal{L}_*(n_1 + \cdots + n_a)$ *given by*

$$\gamma(\phi; \nu_1, \ldots, \nu_a) := \nu_1 \circ_1 (\nu_2 \circ_2 (\cdots \circ_{a-1} (\nu_a \circ_a \nu)))$$

for $\phi \in \mathcal{L}_*(a)$ *and* $\nu_i \in \mathcal{L}_*(n_i)$, $1 \leq i \leq a$, *define on* $\mathcal{L}_*(*)$ *a structure of a nonsymmetric differential graded operad in the monoidal category of differential graded vector spaces.*

Let N be a (graded) $(V, \cdot)$-bimodule, let $d : N \to N$ be a $(V, \cdot)$-linear differential and define $X_k(n) := \{\theta \in \mathrm{Der}_V^k(\bigodot(N));\ \theta(N) \subset \bigodot^n(N)\}$. For $\omega \in X_*(m)$, $\theta \in X_*(n)$ and $1 \leq i \leq n$ let $\omega_N := \omega|_N : N \to \bigodot^m(N)$ and $\theta_N := \theta|_N : N \to \bigodot^n(N)$ be the restrictions. Let then $\omega \circ_i \theta \in X_*(m+n-1)$ be a derivation defined by $(\omega \circ_i \theta)|_N := (\mathbb{1}^{\odot(i-1)} \odot \omega_N \odot \mathbb{1}^{\odot(n-i)}) \circ \theta_N$. Let us extend the differential d to a derivation D of $\bigodot(N)$ and define $\nabla(\theta) := [D, \theta]$.

Lemma 2.5. *The object* $X_*(*) = (X_*(*), \circ_i, \nabla)$ *constructed above is a differential graded comp algebra.*

Proof. Let $\phi \in X_p(a)$, $\psi \in X_q(b)$ and $\nu \in X_r(c)$. The composition $\phi \circ_i (\psi \circ_j \nu)$ is, by definition, given by its restriction $[\phi \circ_i (\psi \circ_j \nu)]_N$ to N as

$$[\phi \circ_i (\psi \circ_j \nu)]_N = (\mathbb{1}^{\odot(i-1)} \odot \phi_N \odot \mathbb{1}^{\odot(b+c-i-1)}) \circ (\psi \circ_j \nu)_N,$$

with $(\psi \circ_j \nu)_N = (\mathbb{1}^{\odot(j-1)} \odot \psi_N \odot \mathbb{1}^{\odot(c-j)}) \circ \nu_N$. This implies that

$$[\phi \circ_i (\psi \circ_j \nu)]_N = (\mathbb{1}^{\odot(i-1)} \odot \phi_N \odot \mathbb{1}^{\odot(b+c-i-1)}) \circ (\mathbb{1}^{\odot(j-1)} \odot \psi_N \odot \mathbb{1}^{\odot(c-j)}) \circ \nu_N.$$

For $i \leq j-1$ we have (taking into the account that $\mathrm{Im}(\phi_N) \subset \bigodot^a(N)$ and $\mathrm{Im}(\psi_N) \subset \bigodot^b(N)$)

$$(\mathbb{1}^{\odot(i-1)} \odot \phi_N \odot \mathbb{1}^{\odot(b+c-i-1)}) \circ (\mathbb{1}^{\odot(j-1)} \odot \psi_N \odot \mathbb{1}^{\odot(c-j)}) =$$
$$= (-1)^{pq} \cdot (\mathbb{1}^{\odot(j+a-2)} \odot \psi_N \odot \mathbb{1}^{\odot(c-j)}) \circ (\mathbb{1}^{\odot(i-1)} \odot \phi_N \odot \mathbb{1}^{\odot(c-i)}),$$

which means that $[\phi \circ_i (\psi \circ_j \nu)]_N = (-1)^{pq} \cdot [\psi \circ_{j+a-1} (\phi \circ_i \nu)]_N$. This is the axiom (4) for $i \leq j-1$.

Similarly, for $j \leq i \leq b+j-1$ we have

$$(\mathbb{1}^{\odot(i-1)} \odot \phi_N \odot \mathbb{1}^{\odot(b+c-i-1)}) \circ (\mathbb{1}^{\odot(j-1)} \odot \psi_N \odot \mathbb{1}^{\odot(c-j)}) =$$
$$= \mathbb{1}^{\odot(j-1)} \odot [(\mathbb{1}^{\odot(i-j)} \odot \phi_N \odot \mathbb{1}^{\odot(b-i+j-1)}) \circ \psi_N] \odot \mathbb{1}^{\odot(c-j)}$$

which means that $[\phi \circ_i (\psi \circ_j \nu)]_N = [(\phi \circ_{i-j+1} \psi) \circ_j \nu]_N$. This is the axiom (4) for $j \leq i \leq b+j-1$. The discussion of the remaining case $i \geq j+b$ is similar. Q.E.D.

Let us consider the special case of the construction above with $N := \uparrow V \oplus \uparrow \mathcal{B}_*(V)$ and $d := 0 \oplus \uparrow d_{\mathcal{B}} \downarrow$.

LEMMA 2.6. *The bigraded subspace $J_*(*)$ of $X_*(*)$ introduced in Definition 2.2 is closed under the operations $\circ_i$ and the differential ∇.*

The proof of the lemma is a straightforward verification. The lemma says that the dg comp algebra structure on $X_*(*)$ restricts to a dg comp algebra structure $(J_*(*), \circ_i, \nabla)$ on $J_*(*)$.

3. More about $\mathrm{Der}^*_V(\bigodot(V, \mathcal{B}_*(V)))$

Let $J_*(*) = (J_*(*), \circ_i, \nabla)$ and $D_{-1} \in \mathrm{Der}^1_V(\bigodot(V, \mathcal{B}_*(V)))$ be as in the previous section.

DEFINITION 3.1. *An infinitesimal deformation of D_{-1} is an element $D_0 \in J_0(2)$ such that $\nabla(D_0) = 0$. An integration of an infinitesimal deformation D_0 is a sequence $\tilde{D} = \{D_i \in J_i(i+2);\ i \geq 1\}$ such that $D := D_{-1} + D_0 + D_1 + \cdots$ satisfies $[D, D] = 0$.*

Let K_n be, for $n \geq 2$, the Stasheff associahedron [7]. It is an $(n-2)$-dimensional cellular complex whose i-dimensional cells are indexed by the set $\mathrm{Br}_n(i)$ of all (meaningful) insertions of $(n-i-2)$ pairs of brackets between n symbols, with suitably defined incidence maps. There is, for any $a, b \geq 2$, $0 \leq i \leq a-2$, $0 \leq j \leq b-2$ and $1 \leq t \leq b$, a map

$$(-,-)_t : \mathrm{Br}_a(i) \times \mathrm{Br}_b(j) \to \mathrm{Br}_{a+b-1}(i+j),\ u \times v \mapsto (u,v)_t,$$

where $(u,v)_t$ is given by the insertion of (u) at the t-th place in v. This map defines, for $a, b \geq 2$ and $1 \leq t \leq b$, the inclusions $\iota_t : K_a \times K_b \hookrightarrow \partial K_{a+b-1}$. It is well-known that the sequence $\{K_n\}_{n\geq 1}$ form a topological operad, see again [7].

Let $CC_i(K_n)$ denote the set of i-dimensional oriented cellular chains with coefficients in $\mathbf{k}$ and let $d_C : CC_i(K_n) \to CC_{i-1}(K_n)$ be the cellular differential. For $\mathbf{s} \in CC_p(K_a)$ and $\mathbf{t} \in CC_q(K_b)$, $p, q \geq 0$, $a, b \geq 2$ and $1 \leq i \leq b$, let $\mathbf{s} \times \mathbf{t} \in C_{p+q}(K_a \times K_b)$ denote the cellular cross product and put

$$\mathbf{s} \circ_i \mathbf{t} := (\iota_i)_*(\mathbf{s} \times \mathbf{t}) \in CC_{p+q}(K_{a+b-1}).$$

Proposition 3.2. *The cellular chain complex $(CC_*(K_*), d_C)$ together with operations $\circ_i$ introduced above forms a differential graded comp algebra.*

The simplicial version of this proposition was proved in [4], the proof of the cellular version is similar. The dg comp algebra structure of Proposition 3.2 reflects the topological operad structure of $\{K_n\}_{n\geq 2}$ mentioned above.

Let c_n be, for $n \geq 0$, the unique top dimensional cell of K_{n+2}, i.e. the unique element of $\mathrm{Br}_n(n+2)$ corresponding to the insertion of no pairs of brackets between $(n+2)$ symbols. Let us define $\mathbf{e}_n \in CC_n(K_{n+2})$ as $\mathbf{e}_n := 1 \cdot c_n$.

The following proposition was proved in [3], see also the comments below.

Proposition 3.3. *The graded comp algebra $CC_*(K_*) = (CC_*(K_*), \circ_i)$ is a free graded comp algebra on the set $\{\mathbf{e}_0, \mathbf{e}_1, \dots\}$.*

Let us recall that the freeness in the proposition above means that for any graded comp algebra $L_*(*) = (L_*(*), \circ_i)$ and for any sequence $\alpha_n \in L_n(n+2)$, $n \geq 0$, there exists a unique graded comp algebra map $f : CC_*(K_*) \to L_*(*)$ such that $f(\mathbf{e}_n) = \alpha_n$, $i \geq 0$.

The proof of Proposition 3.3 is based on the following observation. There is a description of the free graded comp algebra (= free nonsymmetric nonunital operad) on a given set in terms of oriented planar trees. The free comp algebra $\mathcal{F}(\mathbf{e}_0, \mathbf{e}_1, \dots)$ on the set $\{\mathbf{e}_0, \mathbf{e}_1, \dots\}$ has $\mathcal{F}(\mathbf{e}_0, \mathbf{e}_1, \dots)(n)$ = the vector space spanned by oriented connected planar trees with n input edges. Each such a tree T then determines an element of $\mathrm{Br}_i(n)$ where i = the number of vertices of T, i.e. a cell of $CC_{n-i-2}(K_n)$. This correspondence defines a map $\mathcal{F}(\mathbf{e}_0, \mathbf{e}_1, \dots)(n) \to CC_*(K_n)$ which induces the requisite isomorphism of operads.

Let $L = (L_*(*), \circ_i, \nabla)$ be a dg comp algebra. Let us define, for $\phi \in L_p(a)$ and $\psi \in L_q(b)$,

$$\phi \diamond \psi := \sum_{1\leq i\leq b} (-1)^{(a+1)(i+q+1)} \cdot \phi \circ_i \psi \ \text{ and } \ [\phi, \psi] := \phi \diamond \psi - (-1)^{(a+p+1)(b+q+1?)} \cdot \psi \diamond \phi.$$

In [4] we proved the following proposition.

PROPOSITION 3.4. *The operation* $[-,-]$ *introduced above endows* $L^* := \bigoplus_{a-p-1=*} L_p(a)$ *with a structure of a differential graded (dg) Lie algebra,* $L = (L^*, [-,-], \nabla)$.

The construction above thus defines a functor from the category of dg comp algebras to the category of dg Lie algebras. Let us observe that for the comp algebras $X_*(*)$ and $J_*(*)$ this structure coincides with the Lie algebra structure induced by the graded commutator of derivations. We can also easily prove that the elements $\{\mathbf{e}_n\}_{n\geq 0}$ satisfy, for each $m \geq 0$,

$$d_C(\mathbf{e}_m) + \tfrac{1}{2} \textstyle\sum_{i+j=n-1} [\mathbf{e}_i, \mathbf{e}_j] = 0 \tag{5}$$

in the dg Lie algebra $CC(K)^* = (CC(K)^*, [-,-], d_C)$.

PROPOSITION 3.5. *There exists an one-to-one correspondence between integrations of an infinitesimal deformation* D_0 *in the sense of Definition 3.1 and dg comp algebra homomorphisms* $m : CC_*(K_*) \to J_*(*)$ *with* $m(\mathbf{e}_0) = D_0$.

PROOF. Let us suppose we have a map $m : CC_*(K_*) \to J_*(*)$ with $m(\mathbf{e}_0) = D_0$ and define, for $n \geq 1$, $D_n := m(\mathbf{e}_n)$. We must prove that the derivation $D := D_{-1} + D_0 + D_1 + \cdots$ satisfies $[D, D] = 0$. This condition means that

$$\nabla(D_m) + \tfrac{1}{2} \textstyle\sum_{i+j=n-1} [D_i, D_j] = 0 \tag{6}$$

for any $m \geq 0$, which is exactly what we get applying on (5) the Lie algebra homomorphism m.

On the other hand, suppose we have an integration $\{D_n\}_{n\geq 1}$. The freeness of the graded comp algebra $CC_*(K_*)$ (Proposition 3.3) ensures the existence of a graded comp algebra map $m : CC_*(K_*) \to J_*(*)$ with $m(\mathbf{e}_n) = D_n$ for $n \geq 0$. We must verify that this unique map commutes with the differentials, i.e. that $m(d_C(\mathbf{s})) = \nabla(m(\mathbf{s}))$ for any $\mathbf{s} \in CC_*(K_*)$. Because of the freeness, it is enough to verify the last condition for $\mathbf{s} \in \{\mathbf{e}_n\}_{n\geq 0}$, i.e. to verify that $m(d_C(\mathbf{e}_n)) = \nabla(m(\mathbf{e}_n))$ for $n \geq 0$. Expanding $d_C(\mathbf{e}_n)$ using (5) we see that this follows from (6) and from the fact that the map m is a homomorphism of graded Lie algebras. Q.E.D.

PROPOSITION 3.6. *An infinitesimal deformation* $D_0 \in J_0(2)$ *can be integrated if and only if* $[D_0, D_0]|_{\uparrow V} = [D_0, D_0]|_{\uparrow \mathcal{B}_1(V)} = 0$.

PROOF. Standard obstruction theory. Let us suppose that we have an integration $\tilde{D} = \{D_i\}_{i\geq 1}$. Condition (6) with $m = 1$ means that

$$\nabla(D_1) + \tfrac{1}{2}[D_0, D_0] = 0. \tag{7}$$

Because $D_1 \in J_1(3)$, $D_1|_{\uparrow V} = D_1|_{\uparrow \mathcal{B}_1(V)} = 0$ by (iii) of Definition 2.2, therefore $\nabla(D_1)|_{\uparrow V} = \nabla(D_1)|_{\uparrow \mathcal{B}_1(V)} = 0$ and (7) implies that $[D_0, D_0]|_{\uparrow V} = [D_0, D_0]|_{\uparrow \mathcal{B}_1(V)} = 0$.

On the other hand, let us suppose that $[D_0, D_0]|_{\uparrow V} = [D_0, D_0]|_{\uparrow \mathcal{B}_1(V)} = 0$. By the description of $H_0(J_*(3), \nabla)$ as it is given in Proposition 2.3 we see that the homology class of $[D_0, D_0] \in J_0(3)$ is zero, therefore there exists some $D_1 \in J_1(3)$ with $\nabla(D_1) + \frac{1}{2}[D_0, D_0] = 0$.

Let us suppose that we have already constructed a sequence $D_i \in J_i(i+2)$, $1 \leq i \leq N$, such that basic equation (6) holds for any $m \leq N$. The element $\frac{1}{2}\sum_{i+j=N}[D_i, D_j] \in J_N(N+3)$ is a ∇-cycle (this follows from the definition of $\nabla(-)$ as $[D_{-1}, -]$ and the Jacobi identity) and the triviality of $H_N(J_*(N+3))$ (again Proposition 2.3) gives some $D_{N+1} \in J_{N+1}(N+3)$ which satisfies (6) for $m = N+1$. The induction may go on. Q.E.D.

Let G be the subgroup of $\mathrm{Aut}(\bigodot(V, \mathcal{B}_*(V))^*)$ consisting of automorphisms of the form $g = 1\!\!1 + \phi_{\geq 2}$, where $\phi_{\geq 2}$ is a $(V, \cdot)$-linear map satisfying $\phi_{\geq 2}(\bigodot^{i,j}(V, \mathcal{B}_*(V))) \subset \bigodot^{\geq i+2, j}(V, \mathcal{B}_*(V))$ for any i, j.

Let us observe that G naturally acts on the set of integrations of a fixed infinitesimal deformation D_0. To see this, let $\tilde{D} = \{D_i\}_{i \geq 1}$ be such an integration and let us denote, as usual, $D := D_{-1} + D_0 + D_1 + \cdots$. Then $g^{-1}Dg$ is, for $g \in G$, clearly also a $(V, \cdot)$-linear derivation from $\mathrm{Der}^1_V(\bigodot(V, \mathcal{B}_*(V)))$ and $(g^{-1}Dg)(\bigodot^{i,j}(V, \mathcal{B}_*(V))) \subset (\bigodot^{\geq i,j}(V, \mathcal{B}_*(V)))$. We may thus decompose $g^{-1}Dg$ as $g^{-1}Dg = \sum_{k \geq -1} D'_k$ with $D'_k(\bigodot^{i,j}(V, \mathcal{B}_*(V))) \subset \bigodot^{i+k+1, j}(V, \mathcal{B}_*(V))$. We observe that $D'_{-1} = D_{-1}$, $D'_0 = D_0$ and that, from degree reasons, $D'_k(\bigodot(V, \mathcal{B}_*(V))^{i,j}) \subset \bigodot(V, \mathcal{B}_*(V))^{i+k+1, j-k}$. This means that $D'_k \in J_k(k+2)$ for $k \geq 1$. The equation $[g^{-1}Dg, g^{-1}Dg] = 0$ is immediate, therefore the correspondence $(g, \{D_i\}_{i \geq 1}) \mapsto \{D'_i\}_{i \geq 1}$ defines the requisite action. The following proposition shows that this action is transitive.

Proposition 3.7. *Let $\tilde{D}' = \{D'_i\}_{i \geq 1}$ and $\tilde{D}'' = \{D''_i\}_{i \geq 1}$ be two integrations of an infinitesimal deformation D_0. If we denote $D' := D_{-1} + D_0 + \sum_{i \geq 1} D'_i$ and $D'' := D_{-1} + D_0 + \sum_{i \geq 1} D''_i$, then $D' = g^{-1}D''g$ for some $g \in G$.*

Proof. Again standard obstruction theory. As we already observed, for any $g \in G$, $g^{-1}D''g$ decomposes as $g^{-1}D''g = D_{-1} + D_0 + \sum_{i \geq 1}\{g^{-1}D''g\}_i$ with some $\{g^{-1}D''g\}_i \in J_i(i+2)$. Let us suppose that we have already constructed some $g_N \in G$, $N \geq 1$, such that $\{g_N^{-1}D''g_N\}_i = D'_i$ for $1 \leq i \leq N$. We have

$$\nabla(D'_{N+1}) - \tfrac{1}{2}\textstyle\sum_{i+j=N}[D'_i, D'_j] = 0$$

and, similarly,

$$\nabla(\{g_N^{-1}D''g_N\}_{N+1}) - \tfrac{1}{2}\textstyle\sum_{i+j=N}[\{g_N^{-1}D''g_N\}_i, \{g_N^{-1}D''g_N\}_j] = 0.$$

By the induction, the second terms of the above equations are the same, therefore $\nabla(D'_{N+1}) = \nabla(\{g_N^{-1}D''g_N\}_{N+1})$ which means that $D'_{N+1} - \{g_N^{-1}D''g_N\}_{N+1} \in J_{N+1}(N+3)$ is a cycle. The triviality of $H_{N+1}(J_*(N+3))$ (Proposition 2.3) gives some $\phi \in J_{N+2}(N+3)$ such that $D'_{N+1} - \{g_N^{-1}D''g_N\}_{N+1} = \nabla(\phi)$. The element $\exp(\phi) \in G$ is of the form $1\!\!1 + \phi + \phi_{\geq N+3}$ with

$$\phi_{\geq N+3}(\bigodot^{i,j}(V, \mathcal{B}_*(V))) \subset \bigodot^{i+N+3, j}(V, \mathcal{B}_*(V)),$$

therefore $g_{N+1} := \exp(\phi) g_N$ satisfies $\{g_{N+1}^{-1} D'' g_{N+1}\}_i = \{g_N^{-1} D'' g_N\}_i = D'_i$ for $1 \leq i \leq N$ and $\{g_{N+1}^{-1} D'' g_{N+1}\}_{N+1} = \{g_N^{-1} D'' g_N\}_{N+1} + \nabla(\phi) = D'_{N+1}$ and the induction goes on. The prounipotency of the group G assures that the sequence $\{g_N\}_{N \geq 1}$ converges to some $g \in G$ as required.

Q.E.D.

4. Applications to Drinfel'd algebras

In [4] we introduced two $(V, \cdot)$-linear 'coactions' $\lambda : \mathcal{B}_*(V) \to V \odot \mathcal{B}_*(V)$ and $\rho : \mathcal{B}_*(V) \to \mathcal{B}_*(V) \odot V$ as

$$
\begin{aligned}
\lambda(a_0|\cdots|a_{-n+1}) &:= \sum \Delta'(a_0)\cdots\Delta'(a_{-n+1}) \odot (\Delta''(a_0)|\cdots|\Delta''(a_{-n+1})), \text{ and} \\
\rho(a_0|\cdots|a_{-n+1}) &:= \sum (\Delta'(a_0)|\cdots|\Delta'(a_{-n+1})) \odot \Delta''(a_0)\cdots\Delta''(a_{-n+1}).
\end{aligned}
$$

Let us define a derivation $D_0 \in J_0(2)$ by

$$
D_0|_{\uparrow \mathcal{B}_*(V)} := (\uparrow \bigodot \uparrow)(\lambda + \rho)(\downarrow) \text{ and } D_0|_{\uparrow V} := (\uparrow \bigodot \uparrow)(\Delta)(\downarrow).
$$

Proposition 4.1. *The derivation D_0 defined above is an integrable infinitesimal deformation of D_{-1}.*

Proof. To prove that D_0 is an infinitesimal deformation of D_{-1} means to show that $\nabla(D_0) = [D_{-1}, D_0] = 0$. This was done in [4].

The integrability of D_0 means, by Proposition 3.6, that $[D_0, D_0]|_{\uparrow V} = [D_0, D_0]|_{\uparrow \mathcal{B}_1(V)} = 0$. For $v \in V$ we have

$$
\begin{aligned}
[D_0, D_0](\uparrow v) &= (D_0 \circ D_0)(\uparrow v) = D_0(\uparrow \odot \uparrow)(\Delta)(\uparrow v) = \\
&= [((\uparrow \odot \uparrow)(\Delta) \odot \uparrow)(\Delta) - (\uparrow \odot (\uparrow \odot \uparrow)(\Delta))(\Delta)](\uparrow v) = \\
&= (\uparrow \odot \uparrow \odot \uparrow) \circ [(\Delta \odot 1\!\!1)(\Delta) - (1\!\!1 \odot \Delta)(\Delta)(\uparrow v)]
\end{aligned}
$$

which is zero by the quasi-coassociativity (1) and by (2).

Similarly, for $(v) \in \mathcal{B}_1(V)$ we have

$$
D_0^2 = (\uparrow \odot \uparrow \odot \uparrow)[(\Delta \odot 1\!\!1)\lambda - (1\!\!1 \odot \lambda)\lambda - (1\!\!1 \odot \rho)\lambda + (\lambda \odot 1\!\!1)\rho + (\rho \odot 1\!\!1)\rho - (1\!\!1 \odot \Delta)\rho](\downarrow (v))
$$

and (1), (2) again imply that this is zero. Q.E.D.

Definition 4.2. *An integration of the infinitesimal deformation D_0 above is called a homotopy comodule structure.*

Let $\bigodot'(V,\mathcal{B}_*(V)) = \bigodot^{*,1}(V,\mathcal{B}_*(V))$ denote the submodule of $\bigodot^{*,*}(V,\mathcal{B}_*(V))$ with precisely one factor of $\uparrow\mathcal{B}_*(V)$. Let $C^n(A)$ be the set of all degree n homogeneous maps $f : \bigodot'(V,\mathcal{B}_*(V)) \to \bigodot^*(\uparrow V)$ which are both $\bigodot^*(\uparrow V)$ and $(V,\cdot\,)$-linear. Let us define also a degree one derivation d_C on $\bigodot^*(\uparrow V)$ by $d_C|_{\uparrow V} := (\uparrow\odot\uparrow)(\Delta)(\uparrow)$.

Let $\{D_i\}_{i\geq 1}$ be a homotopy comodule structure in the sense of Definition 4.2 and let $D := D_{-1}+D_0+D_1+\cdots$. Define a degree one endomorphism d of $C^*(A)$ by $d(f) := f\circ D+(-1)^n d_C\circ f$. It is easy to show that d is a differential and, following [4], we define the *cohomology of our Drinfel'd algebra A* by $H^*(A) := H^*(C^*(A),d)$.

Proposition 4.3. *The definition of the cohomology of a Drinfel'd algebra does not depend on the particular choice of a homotopy comodule structure.*

Proof. Let $\{D'_i\}_{i\geq 1}$ and $\{D''_i\}_{i\geq 1}$ be two homotopy comodule structures, $D' := D_{-1}+D_0+D'_1+\cdots$ and $D'' := D_{-1}+D_0+D''_1+\cdots$. Let $d'(f) := f\circ D'+(-1)^n d_C\circ f$ and $d'(f) := f\circ D''+(-1)^n d_C\circ f$. Proposition 3.7 then gives some $g\in G$ such that $D'\circ g = g\circ D''$. We see immediately that the map $\Psi : (C^*(A),d') \to (C^*(A),d'')$ defined by $\Psi(f) := f\circ g$ is an isomorphism of complexes. Q.E.D.

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